Use Elementary Column Operations to Calculate the Basis of the Null Space of a Matrix

by Chen Bihong

In linear algebra, the kernel or null space (also nullspace) of a matrix A is the set of all vectors x for which Ax=0. The kernel of a matrix with n columns is a linear subspace of n-dimensional Euclidian space. The dimension of null space of A is called the nullity of A. With traditional method, for calculating the basis of null space of A, we must use row reduction to find a basis for the null space. That is, first use elementary row operations to put A in reduced row echelon form, then interpreting the reduced row echelon form as a homugeneous linear system, determine which of the variables in terms of the free variables, then write equations for the dependent variables in terms of the free variables, then for each free variable xi, choose the vector in the null space for which xi=1 and the remaining free variables are zero then resulting collection of vectors is a basis for the null space of A. In this paper we call it elementary row operations method.

Intruduction

Now I give another method which is called elementary column operations method. It is given by theorem as follows.

Chen Bihong Theorem:

Given a matrix $\mathbf{A}_{m\times n}$, rank($\mathbf{A}$)=$r<n$

$$\begin{pmatrix} \mathbf{A}_{m\times n} \\ \mathbf{E}_n \end{pmatrix} \xrightarrow{\text{Elementary Column Operations}} \begin{pmatrix} \mathbf{B}_{m\times r} & \mathbf{O}_{m\times(n-r)} \\ \mathbf{P}_{n\times r} & \mathbf{Q}_{n\times(n-r)} \end{pmatrix}$$

n-r columns of **Q** are basis of null space of matrix **A**.

Given a m rows n columns matrix A with rank(A)=r, and r less than n, construct a partition metrix by adding identity matrix E below A, then do elementary column operations to it to let A become (B, O) where O is a zero matrix with n-r columns, so the matrix is changed to partition matrix B,O,P,Q , and the all n minul r columns of Q are the basis of null space of matrix A.

Proof this theorem is easy, but I will proof it use some special method as follows to suggest some thinking methods and teaching methods.

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n-r columns of $\mathbf{Q}$ are basis of null space of matrix $\mathbf{A}$.

Pinciple

Old system:

$$\mathbf{Ax=b}$$

New system:

$$\mathbf{PAQy=Pb}$$

where **P** and **Q** are invertible matrix.

We call a usual system of linear equations Ax equal to b as old system, where A is a matrix with m rows and n columns and b is a column vector of n entrays. Given any invertible square matrix P of order m and invertible square matrix Q of order n, change the old system to PAQx equals Pb witch are called new system.

Old system: **$Ax=b$**

New system: **$PAQx=Pb$**

Given a solution kesai of the old system, so

$$A\xi=b$$

and let **$\eta=Q^{-1}\xi$, then η is solution of new system.**

because

$$PAQ\eta=PAQQ^{-1}\xi=PA\xi=Pb.$$

Otherwise, **given a solution η of the new system,**

$$PAQ\eta=Pb$$

Left multiply P-1 on two side of the equation can get

$$AQ\eta=b$$

so $\xi=Q\eta$ is a solution of the old system.

Old system:

$$\mathbf{Ax=b}$$

New system:

$$\mathbf{PAQx=Pb}$$

If ξ is a solution of old system,
then $\eta = \mathbf{Q}^{-1}\xi$ is a solution of new system.
If η is a solution of new system,
then $\xi = \mathbf{Q}\eta$ is a solution of old system.

So the two sets of solutions of old and new systems have relation of one to one and invertible linear transformation. If we select suitable P and Q to let new system have simple form, we can find the new method to calculate solutions of system of linear equations

Standard matrix:

O and E are standard matrix.

Add some zero rows or zero columns to a standard matrix to get a new standard matrix.

Use $\mathbf{D}_{m\times n}$ to express a standard matrix, rank($\mathbf{D}$)=r

Standard system:

$$\mathbf{D}_{m\times n}\mathbf{x}=\mathbf{d}$$

where **D** is standard matrix,

$r(\mathbf{D})=r$.

$\mathbf{d}=(d_1,d_2,\ldots,d_r,d_{r+1},\ldots,d_m)$

Standard system if solutable, it must be

$d_{r+1}=d_{r+2}=\ldots=d_m=0$, if they are exist.

Then system can write as:

x_{r+1} to x_n can be any value.

$$\begin{cases} x_1 = d_1, \\ x_2 = d_2, \\ \vdots \\ x_r = d_r, \\ 0 = 0, \\ \vdots \\ 0 = 0. \end{cases}$$

So the resolution **x** can write as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \\ c_1 \\ c_2 \\ \vdots \\ c_{n-r} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \\ 0 \\ 0 \\ \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 0 \\ 0 \\ \\ 0 \\ 1 \\ 0 \\ \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \gamma + c_1\mathbf{e}_{r+1} + c_2\mathbf{e}_{r+2} + \cdots$$

change the note $\mathbf{x}$ to $\mathbf{y}$, then $\mathbf{Dy}=\mathbf{d}$, its solution are:

$\mathbf{y}=\gamma+c_1\mathbf{e}_{r+1}+c_2\mathbf{e}_{r+2}+\ldots+c_{n-r}\mathbf{e}_n$,

where $\mathbf{e}_i$ is i-th column of $\mathbf{E}_n$.

and old system is

$$\mathbf{Ax}=\mathbf{b}$$

use invertible matrix $\mathbf{P}$ and $\mathbf{Q}$, let $\mathbf{D}=\mathbf{PAQ}$, this means do elementary operations to A.

then old system's solutions are

$\mathbf{x}=\mathbf{Qy}=\mathbf{Q}\gamma+c_1\mathbf{Qe}_{r+1}+c_2\mathbf{Qe}_{r+2}+\ldots+c_{n-r}\mathbf{Qe}_n$

Old system: $\mathbf{Ax}=\mathbf{b}$

New system: $\mathbf{PAQy}=\mathbf{Pb}$ or $\mathbf{Dy}=\mathbf{d}$

Old system's solution:

$$\mathbf{x}=\mathbf{Q}\gamma+c_1\mathbf{Qe}_{r+1}+c_2\mathbf{Qe}_{r+2}+\ldots+c_{n\text{-}r}\mathbf{Qe}_n$$

and

$\mathbf{Qe}_i$ is just the *i*-th column of $\mathbf{Q}$!

$\mathbf{Q}$ is the record of doing elementary column operation to $\mathbf{A}$ to change $\mathbf{A}$ to standard matrix $\mathbf{D}$, so add $\mathbf{E}$ below $\mathbf{A}$, to record the $\mathbf{Q}$

construct matrix

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{E} \end{pmatrix}$$

do elementary operations to **A**'s part to change **A** to standard matrix **D**, then **E** becomes **Q**.

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{E} \end{pmatrix} \to \begin{pmatrix} \mathbf{D} \\ \mathbf{Q} \end{pmatrix}$$

But row operations are not need to do.

Example: Let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & -2 & 2 \\ -1 & 1 & 2 & -2 \end{pmatrix}$$

construct

$$\left(\frac{\mathbf{A}}{\mathbf{E}}\right) = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & -2 & 2 \\ -1 & 1 & 2 & -2 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} \\ \hline \mathbf{E} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & -2 & 2 \\ -1 & 1 & 2 & -2 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[c_4+c_1]{\substack{c_2+c_1 \\ c_3-c_1}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -4 & 4 \\ -1 & 0 & 3 & -3 \\ \hline 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{c_4+c_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -4 & 0 \\ -1 & 0 & 3 & 0 \\ \hline 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a basis of null space of matrix **A**:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Old method:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & -2 & 2 \\ -1 & 1 & 2 & -2 \end{pmatrix} \xrightarrow[r_3+r_1]{r_2-2r_1} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 3 & -3 \end{pmatrix}$$

$$\xrightarrow[\substack{r_3-3r_2 \\ r_1-r_2}]{r_2\div(-4)} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Old method:

$$\mathbf{A}=\begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & -2 & 2 \\ -1 & 1 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

go back to system of linear equations:

$$\begin{cases} x_1 = x_2, \\ x_3 = x_4. \end{cases}$$

Let $x_2=c_1, x_4=c_2$,

c_1, c_2 can be any number.

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Let $x_2 = c_1, x_4 = c_2$, c_1, c_2 can be any number.

Then we get

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1 \\ c_2 \\ c_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Conclusion

After Chen Bihong theorem is discovered, a lot of concepts of linear algebra, like "free variable", "reduced row echelon", "Gaussian elimination", etc. become stupid and will be discarded or less use. Entire textbook of linear algebra should rewrite, and a lot of computer programs to calculate the solution of system of linear equations should be reprogrammed.

Thank you!

Example 1: Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 1 \\ 1 & 3 & 1 & -1 \end{pmatrix}$$

Then

$$\left(\frac{\mathbf{A}}{\mathbf{E}}\right) = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 1 \\ 1 & 3 & 1 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 1 \\ 1 & 3 & 1 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[c_4-3c_1]{\substack{c_2-2c_1 \\ c_3-c_1}} \left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -5 \\ 1 & 1 & 0 & -4 \\ \hline 1 & -2 & -1 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$(-1,0,1,0)^T$ is a basis of nulspace of **A**.

Example 2: For a lunch, a big monk need 3 breads, 3 small monk need 1 bread, 100 monk just eat 100 breads, calculate the number of big monk and small monk.

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Solution: Let x and y is the number of big monk and small monk, so

$$\begin{cases} x + y = 100, \\ 3x + \frac{1}{3} y = 100. \end{cases} \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 100 \\ 3 & \frac{1}{3} & 100 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 100 \\ 3 & 1/3 & 100 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow[c_3-100c_1]{c_2-c_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 3 & -8/3 & -200 \\ \hline 1 & -1 & -100 \\ 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[c_3\times(-1)]{c_3-\frac{600}{8}c_2} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 3 & -8/3 & 0 \\ \hline 1 & -1 & 25 \\ 0 & 1 & 75 \end{array}\right)$$

so, $x=25$, $y=75$.

Example 3: Determine whether the vectors $\boldsymbol{v}_1$=(1 2 3), $\boldsymbol{v}_2$=(1 2 -1), $\boldsymbol{v}_3$=(3 -1 0), and $\boldsymbol{v}_4$=(2 1 2) form a basis of F^3. If not, choose, if possible, a basis of F^3 consisting of vector of the given set of vectors.

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Solution: Construct matrix **A**:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 3 & 2 \\ 2 & 2 & -1 & 1 \\ 3 & -1 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{E} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 & 2 \\ 2 & 2 & -1 & 1 \\ 3 & -1 & 0 & 2 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[c_4 - 2c_1]{\substack{c_2 - c_1 \\ c_3 - 3c_1}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -7 & -3 \\ 3 & -4 & -9 & -4 \\ \hline 1 & -1 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -7 & -3 \\ 3 & -4 & -9 & -4 \\ \hline 1 & -1 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{c_4-(\frac{3}{7})c_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -7 & 0 \\ 3 & -4 & -9 & -\frac{1}{7} \\ \hline 1 & -1 & -3 & -\frac{5}{7} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{7} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -7 & 0 \\ 3 & -4 & -9 & -\frac{1}{7} \\ \hline 1 & -1 & -3 & -\frac{5}{7} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{7} \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{c_4 - \frac{1}{28}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -7 & 0 \\ 3 & -4 & -9 & 0 \\ \hline 1 & -1 & -3 & -\frac{19}{28} \\ 0 & 1 & 0 & -\frac{1}{28} \\ 0 & 0 & 1 & -\frac{3}{7} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is a basis of F^3, and

$$v_4 = \frac{19}{28} v_1 + \frac{1}{28} v_2 + \frac{3}{7} v_3$$

Example 4: Get the general resolution of equation $x_1+x_2+2x_3=1$.

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resolution: construct matrix as follows:

$$\left(\begin{array}{ccc:c} 1 & 1 & 2 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\substack{c_4-c_1 \\ c_4\times(-1)}]{\substack{c_2-c_1 \\ c_3-2c_1}} \left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ \hline 1 & -1 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc:c} 1 & 1 & 2 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\substack{c_4 - c_1 \\ c_4 \times (-1)}]{\substack{c_2 - c_1 \\ c_3 - 2c_1}} \left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ \hline 1 & -1 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + k_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, (k_1, k_2 \in F)$$

$$x_1 + x_2 + 2x_3 = 1$$